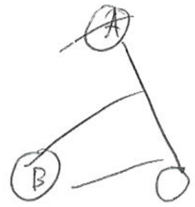
Approximation and Online algorithms with Applications

~~#3.~~

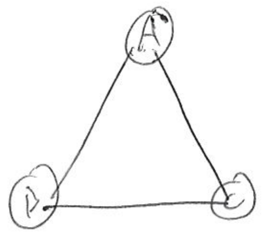
#4

Vertex Cover.

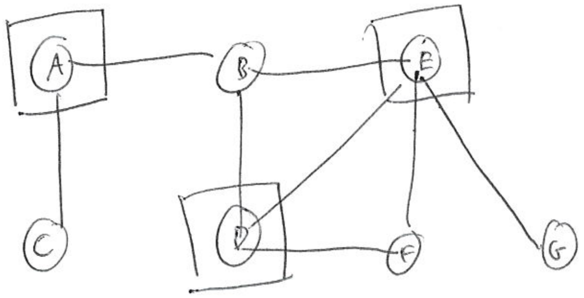
- We wanto to track all ~~information~~ communication in the internet.
- We will hire a number of spies. that report their communication with their friend.
- We want to minimize the number of spies.



Example



- We cannot capture all communications with one person.
- ~~All~~ We need 2 spies.



Smallest set ~~of~~ of spies.

$\{A, D, E\}$.

Optimization Model

Input: A social network (V, E)

Set of persons

set of relationship.

Ex $V = \{A, B, C, D, E, F, G\}$

$E = \{\{A, C\}, \{A, B\}, \{B, D\}, \{B, E\}, \{D, E\}, \{D, F\}, \{E, F\}, \{E, G\}\}$.

Output: $S \subseteq V$. Objective Function: Minimize $|S|$

Constraint: For all $e \in E$, $e \cap S \neq \emptyset$

Ex $S = \{A, D, F\}$, $e = \{A, C\}$, $e \cap S = \{A\} \neq \emptyset$

Reformulate our problem

Output: $x_1, \dots, x_n$: $x_i = \begin{cases} 1 & \text{: we select } i \text{ as a spy} \\ 0 & \text{: otherwise} \end{cases}$

Objective Function Minimize $\sum_{i=1}^{n} x_i$

Constraint: For all $\{u, v\} \in E$, x_u or $x_v = 1$

$$x_u + x_v \geq 1$$

Ex Minimize $x_A + x_B + x_C + \dots + x_G$

such that

$x_A + x_C \geq 1$ $\quad$ $x_D + x_F \geq 1$

$x_A + x_B \geq 1$ $\quad$ $x_E + x_F \geq 1$

$x_B + x_D \geq 1$ $\quad$ $x_E + x_G \geq 1.$

$x_B + x_E \geq 1$

$x_D + x_E \geq 1$

$$\begin{bmatrix} 1 \cdot x_A + 0 \cdot x_B + 1 \cdot x_C + 0 \cdot x_D + \dots + 0 \cdot x_G \\ 1 \cdot x_A + 1 \cdot x_B + 0 \cdot x_C + 0 \cdot x_D + \dots + 0 \cdot x_G \\ 0 \cdot x_A + 1 \cdot x_B + 0 \cdot x_C + 1 \cdot x_D + \dots + 0 \cdot x_G \\ \vdots \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix} x_A \\ x_B \\ x_C \\ x_D \\ x_E \\ x_F \\ x_G \end{bmatrix}
\geq
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

x : ~~vec~~ n-dimensional vector

b : m-dimensional vector with all elements equal to one.

A : $m \times n$ matrix $\quad a_{i,j} = \begin{cases} 1 & j \in e_i \\ 0 & \text{otherwise.} \end{cases}$

When $E = \{e_1, \dots, e_m\}$ and $V = \{1, \dots, n\}$

c : n-dimensional vector with all elements equal to 1.

$$c^t \cdot x = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^t \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i \longrightarrow \text{objective value.}$$

~~Ex~~: Reformulate the problem (again)

<u>Input</u> : Matrix A, vectors b, c.

<u>Output</u> : vector x <u>all elements ~~is~~ are 0, 1</u>

<u>Objective Function</u> : Min. $c^t \cdot x$

<u>Constraint</u> : $Ax \geq b$.

Linear Programming (?)
[but, all element has to be 0, 1]

NP-Hard — can not be solved.

Let make the problem easier

Output : vector x^*, all elements are between 0 to 1

(linear programming can solve when each variable is in a specific range) $\longrightarrow$ Optimal ~~value~~ OPT_R Optimal solution $x' = \begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix}$

Approximation Algorithm

```
Solve the problem that x*_i ∈ [0,1] using linear programming.
for i = 1 to n:
    if x_i < 1/2 :
        x'_i = 0
    Else
        x'_i = 1
return [x'_1 ... x'_n]
```

$$\sum_{i=1}^{n} x_i = OPT_R$$

Theorem The algorithm is 2-approximation algorithm

~~OPT SOL~~

$SOL \le 2 \cdot OPT$

Proof

$$\sum_{i=1}^{n} x_i = OPT_R \le OPT$$

we have more choice for the real number case

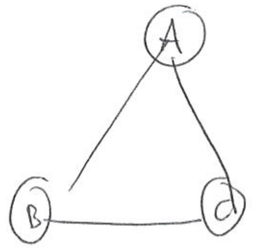
$$x'_i \le 2 \cdot x_i$$

$$SOL = \sum_{i=1}^{n} x'_i \le 2 \cdot \sum_{i=1}^{n} x_i \le 2 \cdot OPT$$

$$SOL \le 2 \cdot OPT.$$

∎

Ex



Solution from linear programming

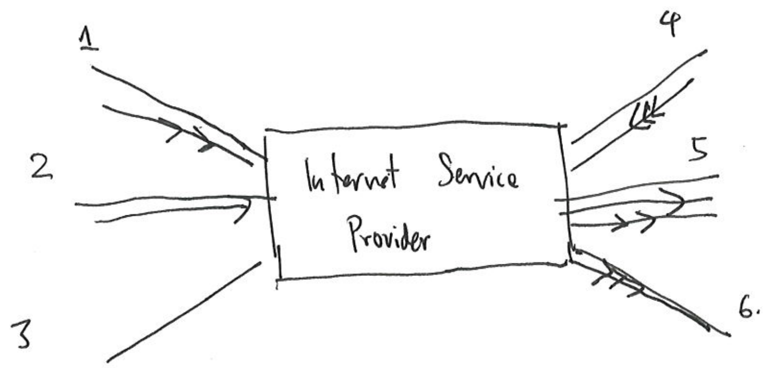
$x_A = x_B = x_C = \frac{1}{2}$ $\qquad \cancel{\sum} \; x_A + x_B + x_C = \frac{3}{2}$

$x'_A = x'_B = x'_C = 1.$ $\qquad x'_A + x'_B + x'_C = 3 = SOL$

optimal number of spies $= 2 = OPT$

Anomalies Detection Using Passive Probes.

[Agrawal, Naidu, and Rastogi INFOCOM '07]



- We want to set passive probes ~~to~~ on communication lines.
- We want to monitor all communications with smallest # probes.

Ex We need 2 probes at links 4, 5.

Optimization Models

Input: # Communication links n. $\{1, \dots, n\}$.

communications m.

$\langle i_1, o_1 \rangle, \langle i_2, o_2 \rangle, \dots, \langle i_m, o_m \rangle$.

i_j: in-link for communication j.

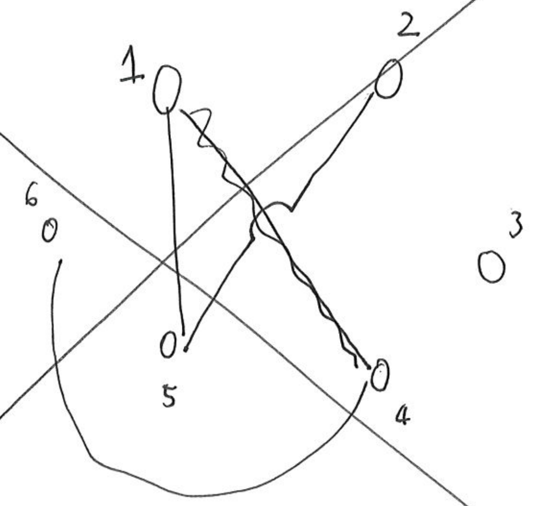
o_j: out-link for communication j.

Output: $S \subseteq \{1, \dots, n\}$ [positions for probes]

Objective Function: Minimize $|S|$.

Constraint: For all communications $\langle i_j, o_j \rangle$, $i_j \in S$ or $o_j \in S$.

See the problem as a graph:



Select nodes such that all comm

See the problem as a social network

~~Prob~~ communication ~~link~~ link → person

probes → spies.

communication → communication.

- Select persons as ~~a~~ spies to spy all communications.

⇓

Select communication link to set up probes to

monitor all communication.

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vertex cover = our optimization Model

2-approximation algorithm for vertex cover ⇒ 2-approximation algorithm for our optimization model.

~~From~~ Until now,

2 main techniques

1. greedy algorithm
   - greedy algorithm + α
   - Knapsack problem.
2. deterministic rounding
   - Linear programming. + some modifications on linear programming solution.
   - Vertex Cover Problem